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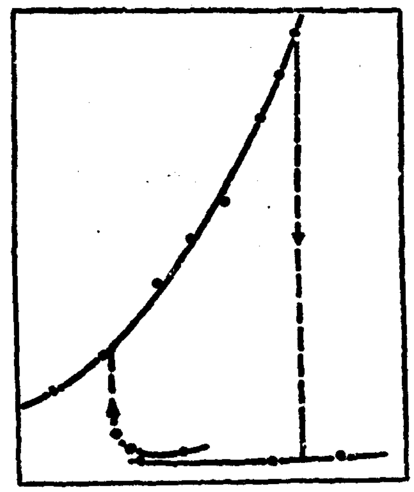

PROJECT THEMIS

Vibration and Stability of Military Vehicles

2

Report No. 17

Response of Systems to

Random Exitation

- A Review -

W. F. Ames

Department of Mechanics and Hydraulics
University of Iowa
Iowa City, Iowa

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RESPONSE OF SYSTEMS TO RANDOM EXCITATION

W. F. Ames
Department of Mechanics and Hydraulics
University of Iowa
Iowa City, Iowa

1. Introduction

The dynamic environments to which weapon-vehicle systems, such as those modeled by Andrews [1], are subjected include steady state harmonic excitation, shock and random excitation. These occur as a result of external stimuli such as atmospherically borne disturbances (wind, wake acoustic noise, rotor tip vortex loading) and through various weapon-vehicle interactions. In the problems of this project random excitation is the rule rather than the exception. Thus this report summarizes the probability techniques necessary for and their application to the development of analytic methods for obtaining the response of linear elastic structures to certain classes of random excitation.

This random vibration response analysis employs the normal modes of a lumped parameter representation of a complex system. The random forcing functions at each mode (in terms of expected value and power spectral density) are transformed to a set of modal forcing functions. Then the response of each mode to a random forcing function can be obtained using the modal transfer functions. Finally the modal responses are transformed back to the physical plane. The results are the statistical expected values (mean, root mean square, power spectral density) of the displacements, velocities and accelerations of the physical system.

2. Basic Probability Concepts

(a) Random Variable

Let x be a real number resulting from a measurement and define an event E as "x is less than or equal to X" where X is a fixed real number. Let the number of trials, out of the first N, in which E is observed, be n_N. If the relative frequency n_N/N tends to a limit, whatever the value of &, that is

$$\Pr [x \leq X] = \lim_{N \to \infty} \frac{n_N}{N} \tag{1}$$

exists for every real number X, then x is called a random variable.

Of the multiplicity of random variable classes two are presently of considerable use. These are discrete and continuous random variables. Our interest herein will be confined to the continuous random variable which we conceive as having the possibility of taking any value over some interval. More precisely we want to be able to evaluate the probability

$$\Pr [X_1 < x \leq X_2] = F(X) \Big|_{x = X_2} - F(x) \Big|_{x = X_1} \tag{2}$$

and we say x is a continuous random variable if F(x) is continuous. Note that since F(x) is continuous then $\lim_{X_1 \to X_2} \Pr [X_1 < x \leq X_2] = 0$, that is continuous random variables have the "odd" property that the probability of taking any one specified value is zero.

The function F(x) is called a cumulative distribution function. These functions have the following common properties (a) $0 \leq F(x) \leq 1$;

(b) F is a nondecreasing function of x; (c) (almost all)

$\lim_{x\to\infty} F(x) = 1$, $\lim_{x\to-\infty} F( ) = 0$.

If the function F(x) is differentiable we define the probability density function p(x) by

$$p(x) = \frac{d}{dx}[F(x)] \tag{3}$$

and note that p(x), like F(x), represents a property of the random variable x. Indeed without knowledge of p(x) (or F(x)) the random variable is computationally essentially useless.

From the definition of p(x) it follows that

$$\Pr[X_1 < x \leq X_2] = \int_{X_1}^{X_2} p(x)\,dx \tag{4}$$

where $X_1 < X_2$

If x and y are two random variables the probabilities $\Pr[x \leq X]$ and $\Pr[y \leq Y]$ each exist. If $\Pr[x \leq X, y \leq Y]$ exists for all X and all Y, the joint cumulative distribution function of x and y is defined by the equation

$$F(x,y) \Big|_{x = X,\, y = Y} = \Pr[x \leq X, y \leq Y]. \tag{5}$$

If both x and y are continuous variables we define their joint probability density function p (x,y) by

$$\iint_R p(x,y)\,dx\,dy = \Pr[(x,y) \text{ in } R]. \tag{6}$$

Here R is any simply connected region of the x, y plane and

$$p(x,y) = \frac{\partial^2 F(x,y)}{\partial x\,\partial y} \tag{7}$$

(b) Basic Properties

Two random variables x and y are mutually independent if the distribution of values of x is unaffected by the value of y, and vice versa. As a consequence of this one finds that

$$F(x,y) = F(x)\ F(y), \quad p(x,y) = p(x)\ p(y) \tag{8}$$

for independent random variables x and y. This "product" property greatly simplifies the computational details for independent processes.

One of the principal features of a random process is its non-repeatability. Therefore, for example, a time history of the response of a vehicle to a random excitation is not very meaningful. Of more interest is the probability of the severity of the structural response to a given input occurring throughout the time interval of interest. For a random process we seek the mean or expected value of the response. If x is a continuous random variable with probability density function p(x) the expected value is defined by the equation

$$E(x) = \int_A^B x\ p(x)\ dx \tag{9}$$

where A and B define the range of values that x can assume. This range is often $(-\infty, \infty)$ or $(0, \infty)$. It is clear that E(x) is not a function of x!

If f(x) is integrable the expected value of f(x) is defined as

$$E[f(x)] = \int_A^B f(x)\ p(x)\ dx. \tag{10}$$

Some elementary but useful properties are immediately obvious. Thus

a) if α is constant $E[\alpha f(x)] = \alpha E[f(x)]$

and $E[f(x) + \alpha] = E[f(x)] + \alpha$

b) $$E[\sum_{j=1}^{n} f_j(x)] = \sum_{j=1}^{n} E[f_j(x)] \qquad (11)$$

In particular we employ the results

$$p(x) = \int_{A_y}^{B_y} p(x,y)\,dy, \quad p(y) = \int_{A_x}^{B_x} p(x,y)\,dx. \qquad (12)$$

Some less obvious properties include the following:

c) if the x_i, $i = 1, 2, \ldots, n$ are mutually independent

then $E[\prod_{i=1}^{n} x_i] = \prod_{i=1}^{n} E(x_i)$.

d) $E[(x-A)^r]$ is the rth moment of x about A. If $A = 0$, $E(x^r)$ is the rth moment about zero. If $A = E(x)$ then $E[(x-A)^r] = \mu_r(x)$ is the rth central moment. In terms of this notation and $E(x)$ we have

$$\mu_2 = \text{var}(x) = \sigma^2(x) = E[(x - E(x))^2]$$
$$= E[x^2] - [E(x)]^2 \qquad (13)$$

e) μ_2 measures the "variability of x".

μ^3 $\mu_3/\mu_2^{3/2}$ measures the skewness of the distribution of x.

μ_4/μ_2^2 measures the kurtosis of the distribution

Additional definitions will be given where needed in the discussion. Details of analysis and background are available in Feller [2] or Parzen [3].

3. **Multi-degree of freedom System**

The governing equations of a linear lumped parameter physical system are

$$M\ddot{X} + C\dot{X} + KX = F(t) \qquad (14)$$

where

$M = [M_{ji}]$ is the matrix of masses and inertias,

$C = [C_{ji}]$ is the matrix of dissipation,

$K = [K_{ji}]$ is the stiffness matrix,

$X = [X_i]$ is the displacement column vector,

$F = [F_j]$ is the column vector of forcing functions, and

$i, j = 1, 2, \ldots, n.$

As is well known these equations can be decoupled with the result that

$$X = \Phi x \qquad (15)$$

where

$\Phi = [\phi_{ij}]$ is the matrix of normal modes (each mode is a column), and $x = [x_j]$ is the column vector of normal coordinates. Upon introducing Eq. (15) into Eq. (14) and premultiplying by Φ^T we have

$$\Phi^T M\Phi\ddot{x} + \Phi^T C\Phi\dot{x} + \Phi^T K\Phi x = \Phi^T F \qquad (16)$$

where $\Phi^T M\Phi$ and $\Phi^T K\Phi$ are diagonal and $\Phi^T C\Phi$ is diagonal if the damping forces are proportional to either stiffness or mass. In such cases the equation of motion, Eq. 14, can be written in the uncoupled form

$$m\ddot{x} + c\dot{x} + kx = f \qquad (17)$$

where m, c and k are diagonal matrices and $f = \Phi^T F$.

The response of an n degree of freedom system becomes the problem

of finding the solutions x_j of the n uncoupled linear differential equations. If the individual responses are desired it is debatable whether the computation should be carried out in this form or in the original form. Often knowledge of the normal modes is highly desirable, in which case the computation as outlined here is required.

4. Response of Single Normal Mode to Deterministic Forcing

The equation for the jth normal mode x_j is

$$\ddot{x}_j + 2\alpha_j \omega_{nj} \dot{x}_j + \omega_{nj}^2 x_j = f_j (t)/m_j \tag{18}$$

where

$$2\alpha_j = c_j/ (k_j m_j)^{1/2}, \ \omega_{nj}^2 = k_j/m_j.$$

Next a (transfer) function H_j relating the modal forcing function to the modal displacement x_j is sought in the form

$$x_j \quad H_j f_j (t). \tag{19}$$

If $f_j (t) = e^{i\omega t}$ then

$$x_j = \frac{m_j^{-1}}{[\omega_{nj}^2 - \omega^2 + 2i\alpha_j \omega\omega_{nj}]} e^{i\omega t} = H_j (\omega) e^{i\omega t}. \tag{20}$$

The utility of $H_j (\omega)$ can be generalized to any forcing function that can be obtained as the superposition of a number of harmonic forcing functions. Thus if $F_j (\omega)$ is one of the components of the Fourier Series representation of $f_j (t)$ defined by

$$F_j (\omega) = \int_{-\infty}^{\infty} f_j (t) e^{-i\omega t} dt \tag{21}$$

then the response to $F_j (\omega)$ is

$\bar{x}_j(\omega) = H_j(\omega) F_j(\omega)$ and the total response is

$$x_j(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \bar{x}_j(\omega)\, e^{i\omega t}\, d\omega$$

$$= \frac{1}{2\pi}\int_{-\infty}^{\infty} H_j(\omega)\, e^{i\omega t}\, d\omega \int_{-\infty}^{\infty} f_j(\tau)\, e^{-i\omega t}\, d\tau \qquad (22)$$

The function H (ω) is also the vehicle used to find the response to a random excitation. Those relationships are mathematically easier to perform by a superposition of elemental solutions in the time domain rather than in the frequency domain. The relation between transfer functions in the time and frequency domains is obtained as follows: Let $\delta(t-\tau)$ be the unit impulse applied at $t = \tau$ and $h_j(t-\tau)$ be the impulse response function. Then the total response to a continuous forcing function $f_j(t)$ is

$$x_j(t) = \int_{-\infty}^{\infty} f_j(\tau)\, h_j(t-\tau)\, d\tau \qquad (23)$$

Also, we know from the linear theory that the impulse response function h_j and $H_j(\omega)$ are related through

$$h_j(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} H_j(\omega)\, e^{i\omega t}\, d\omega$$

$$H_j(\omega) = \int_{-\infty}^{\infty} h_j(t)\, e^{-i\omega t}\, dt \qquad (24)$$

that is they are Fourier and inverse Fourier transforms of one another. This is the desired relationship between transfer functions in the time and frequency domains, assuming these transforms exist.

Some useful transfer functions are listed here:

(a) Force input, displacement response

$$H_j(\omega) = \frac{m_j^{-1}}{[\omega_{nj}^2 - \omega^2 + 2i\zeta_j\,\omega_{nj}\,\omega]} \qquad (25)$$

(b) Force input, acceleration response

$$H_j(\omega) = \frac{-m_j^{-1}\omega^2}{[\omega_{nj}^2 - \omega^2 + 2i\alpha_j\omega_{nj}\omega]} \tag{26}$$

(c) Acceleration input, displacement response

$$H_j(\omega) = \frac{-1}{[\omega_{nj}^2 - \omega^2 + 2i\alpha_j\omega_{nj}\omega]} \tag{27}$$

(d) Acceleration input, acceleration response

$$H_j(\omega) = \frac{\omega^2}{[\omega_{nj}^2 - \omega^2 + 2i\alpha_j\omega_{nj}\omega]} \tag{28}$$

5. Response of a Single Normal Mode to Random Forcing

Since a random vibration is not repeatable a time history of the response of a structure to a random excitation is not very useful. Of more interest is the probability of the severity of the structural response to a given input during a time interval of interest. Knowing the relationship between input and output, $H(\omega)$ or $h(t)$, we can calculate the statistical properties of the response if we know those of the input.

For a random process we seek the expected value of the response assuming we know the expected value of the input. Thus we seek, from Eq. (23), with $\theta = t - \tau$

$$E[x_j(t)] = E[\int_{-\infty}^{\infty} f_j(t-\theta)\, h_j(\theta)\, d\theta]$$

$$= \int_{-\infty}^{\infty} E[f_j(t-\theta)]\, h_j(\theta)\, d\theta \tag{29}$$

where f_j is the only random variable.

We now define several terms which are useful in describing random processes. A stationary process x(t) is one having statistics which do not change with time. If its probability density and all higher order densities are independent of time, we call the process strictly stationary. An ergodic process is a random process for which time averages

$$\overline{g(x)} = \lim_{T\to\infty} \frac{1}{T} \int_0^T g[x(t)]\, dt \tag{30}$$

and ensemble averages (another name for expected values) are equal. All ergodic processes are stationary but not conversely.

If our random process is stationary then it follows from Eq. (29) that

$$E[x_j(t)] = E[f_j(t)] \int_{-\infty}^{\infty} h_j(\theta)\, d\theta. \tag{31}$$

By setting $\omega=0$ in Eq. (24) the value of the integral in Eq. (31) can be evaluated to obtain

$$E[x_j(t)] = H_j(0)\, E[f_j(t)] \tag{32}$$

This becomes

$$E[x_j(t)] = \frac{1}{m_j \omega_{nj}^2} E[f_j] \tag{33}$$

for displacement response to force input, from Eq. (25).

The average value of the product of a function of time with the same function displaced τ sec is called the autocorrelation function

$$R(\tau) = \overline{x(t)\ x(t+\tau)} = \lim_{T\to\infty} \int_0^T x(t)\ x(t+\tau)\ dt \tag{34}$$

to <u>distinguish</u> it from the <u>crosscorrelation</u> function $\overline{g_1(t)\ g_2\ (t+\tau)}$. $R(\tau)$ is particularily important because it forms the link with the frequency - component methods of description.

For an ergodic process we can also write $R(\tau)$ as an ensemble average (expected value) called the <u>covariance function</u>

$$R(\tau) = E[x(t)\ x(t+\tau)]. \tag{35}$$

For an ergodic process $R(\tau)$ has the following properties: (a) $R(0) = \overline{x^2}$; (b) $R(-\tau) = R(\tau)$, that is R is an even function; (c) $R(0) \geq |R(\tau)|$ for all τ.

The relationship between $R(\tau)$ and the frequency - component description of a random function (the <u>spectral density</u>) is developed in the same manner as that relating $h_j(t)$ and $H_j(\omega)$ in Section 4. A sufficient condition for the existence of the Fourier transform of $R(\tau)$ is

$$\int_{-\infty}^{\infty} |R(\tau)|\quad d\tau < \infty$$

The Fourier transform $S(\omega)$ of the autocorrelation function of an ergodic process is called the <u>spectral density</u> defined via

$$S(\omega) = \int_{-\infty}^{\infty} R(\tau)\ e^{-i\omega\tau}\ d\tau \tag{36}$$

(compare with Eq. (24))

and the inversion of Eq. (36) is

$$R(\tau) = \frac{1}{2\pi} \int_{-\infty}^{\infty} S(\omega)\ e^{i\omega\tau}\ d\omega \tag{37}$$

From Eq. (37) it is clear that the mean square value of the random variable

is related to the spectral density by

$$\overline{[x(t)]^2} = R(0) = \frac{1}{2\pi}\int_{-\infty}^{\infty} S(\omega)\,d\omega. \tag{38}$$

A number of properties of $S(\omega)$ are deducible from its definition: (a) $S(\omega)$ is even; (b) $S(\omega)$ is real and positive; (c) the average power dissipated in a one ohm resistor by those frequency components of a voltage $x(t)$ lying in a band between ω and $\omega + d\omega$ is $2S(\omega)\,d\omega$ (the units of $S(\omega)$ are power/cycle/second).

If $S_j(\omega)$ is the spectral density of the random forcing function $f_j(t)$ of Eq. (17) then it follows (see Aseltine [4]) that

$$S_{x_j}(\omega) = |H_j(\omega)|^2\, S_j(\omega) \tag{39}$$

For an ergodic process x_j it follows from the previous remarks that <u>the mean square response</u> is

$$\begin{aligned} E[x_j^{\,2}] &= \overline{[x_j(t)]^2} = R(0) \\ &= \frac{1}{2\pi}\int_{-\infty}^{\infty} S_{x_j}(\omega)\,d\omega \\ &= \frac{1}{2\pi}\int_{-\infty}^{\infty} |H_j(\omega)|^2\, S_j(\omega)\,d\omega. \end{aligned} \tag{40}$$

The values $E[x_j]$, Eq. (32), and the root mean square response $(x_j)_{rms} = \{E[x_j^{\,2}]\}^{1/2}$ define the <u>average</u> properties of the response. The variation of the response from these average values is given by the <u>variance</u>, Eq. (13),

$$\sigma^2(x_j) = E[x_j^{\,2}] - \{E[x_j]\}^2,$$

whose square root is the <u>standard</u> deviation.

6. Probability Distributions

The complete description of a random process requires the selection of a probability distribution. This is usually done by prescribing the probability density function $p(x)$. For the vibration environment of vehicles the actual distribution could be determined by extensive processing of recorded operational data coupled with investigations of the sources of excitation. Investigations of these types have not yet led to definite general conclusions on the nature of the probability distributions. For the purpose of most structure and vehicle vibration tests and responses to random vibration the Gaussian (Normal) distribution is most commonly chosen. Its density function is

$$p(x) = \frac{1}{\sigma\sqrt{2\pi}} \exp[-(x - m)^2/2\sigma^2] \tag{41}$$

where $m = E[x]$ and σ^2 is the variance.

A distribution which has been extensively used in the past two decades is the Weibull distribution (see Johnson and Leone [5], p.112)

$$p(x) = \frac{c}{b} \left(\frac{x-a}{b}\right)^{c-1} \exp[-(\frac{x-a}{b})^c] \tag{42}$$

for $x \geq a$, $b>0$, $c>0$. The cumulative distribution is

$$F(x) = 1 - \exp[- (\frac{x-a}{b})^c]. \tag{43}$$

Note that if $a=0$, $c=1$ the Weibull distribution includes the exponential distribution

$$p(x) = \theta \exp[- \theta x], \theta = 1/b. \tag{44}$$

The general applicability of the Gaussian distribution follows from the Central Limit Theorem. One of many forms of that deMoivre-Laplace result is as follows: "Let the random variable x be distributed with mean μ and variance σ^2 (but with density function unknown). Then the distribution of the sample mean $\bar{x}$ is closely approximated by the Gaussian distribution with mean μ and variance σ^2/n when n is large." (Feller [2]) Thus a process will be approximately Gaussian whenever the process results from the superposition of a large number of sub-processes in which no single sub-process dominates all others.

If the excitation random process $f_j(t)$ has a Gaussian distribution then the response $x_j(t)$ is also Gaussian. With the knowledge of Gaussian response and the resulting statistics for $x_j(t)$ known the probability that the response is below any desired level can be calculated from standard tables. For example, the probability that the absolute value of the response will not exceed the standard deviation σ is .68, that it will not exceed 2σ is .95 and that it will not exceed 3σ is .999.

7. Typical Calculations

We turn now to the evaluation of the mean response $E[x_j]$, and the mean square response $E[x_j^2]$, for several typical forcing functions. Commonly, this input is specified in terms of its frequency limits, spectral density, and mean value.

Mean Value of Response

The mean value of the response $x_j(t)$ as a function of the mean value of the input has been previously established as Eq. (32). Thus we have

a) mean displacement response to a force input:

$$E[x_j(t)] = \frac{1}{m_j\ \omega_{nj}^2}\ E[f_j(t)] \tag{45}$$

b) mean displacement response to an acceleration input:

$$E[x_j(t)] = \frac{-1}{\omega_{nj}^2} E[A_j(t)] \tag{46}$$

c) mean acceleration response to either a force or acceleration input:

$$E[\ddot{X}_j] = 0. \tag{47}$$

<u>Mean Square Response</u>

If ω_1 and ω_2 are the bandwidth frequency limits for a particular excitation function then the general expression for the mean square response, from Eq. (40), becomes

$$E[x_j^2] = \frac{1}{2\pi} \int_{\omega_1}^{\omega_2} |H_j(\omega)|^2 S_j(\omega)\, d\omega \tag{48}$$

For the case $S_j(\omega) = S_{oj}$ = constant (lbs^2/cps)

$$E[x_j^2] = \frac{S_{oj}}{2\pi m_j^2} \int_{\omega_1}^{\omega_2} \frac{d\omega}{(\omega_{nj}^2 - \omega^2)^2 + 4\alpha_j^2 \omega_{nj}^2 \omega^2} \tag{49}$$

is the mean square displacement response to a constant force input.

If the excitation is of the form $S_j(\omega) = A_{oj}\left(\frac{\omega}{\omega_1}\right)^{B/3}$,

$\omega_1 \leq \omega \leq \omega_2$ (lbs^2/cps) then

$$E[\ddot{X}_j^2] = \frac{1}{2\pi} \frac{A_{oj}}{m_j^2} \int_{\omega_1}^{\omega_2} \frac{\omega^4 \left(\frac{\omega}{\omega_1}\right)^{B/3} d\omega}{\omega^4 + 2\omega_{nj}^2 (\alpha_j^2 - 1)\, \omega^2 + \omega_{nj}^4} \tag{50}$$

is the mean square acceleration response to a varying force input.

Similar expressions are obtainable for other mean square responses. The integrals can be evaluated in terms of elementary functions.

8. Response of a Complex Structure to Random Excitation

The solutions of the random responses of each normal mode must now be combined to yield the random response of the multi-degree of freedom system. Modal (f_j) and physical forcing functions (F_j) are related by the equation (see Eq. (16))

$$f_j = \sum_{i=1}^{p} \phi_{ji} F_i \tag{51}$$

where ϕ_{ji} are the elements of ϕ^T. Thus we have

$$E[f_j] = \sum_{i=1}^{p} \phi_{ji} E[F_i] \tag{52}$$

that is the mean value of the modal forcing function is directly computable in terms of the mean values of the physical forcing functions. Equation (52) again follows from the linearity of the expectation operator.

The relationship between the physical and modal spectral densities, used in our previous work, is more complex. From specific forms of Eq. (38) the relationship between input force and input spectral density is

$$E[F_i^2(t)] = \frac{1}{2\pi} \int_{\omega_1}^{\omega_2} \bar{S}_i (\omega)\, d\omega \tag{53}$$

and in the modal plane

$$E[f_j^2(t)] = \frac{1}{2\pi} \int_{\omega_1}^{\omega_2} S_j (\omega)\, d\omega \tag{54}$$

Since

$$f_j^{\,2} = \sum_{r,\ s=1}^{n} \phi_{jr}\ \phi_{js}\ F_r\ F_s \tag{55}$$

then

$$E[f_j^{\,2}] = \sum_{r=1}^{n} \phi_{jr}^{2}\ E[F_r^{\,2}] + \sum_{r=1}^{n} \sum_{s=1}^{n} \phi_{jr} \phi_{js}\ F_r\ F_s \tag{56}$$

Using Eqs. (55) and (56) it follows from Eq. (40), for ergodic processes, that

$$S_j(\omega) = \sum_{r,s}^{n} \phi_{jr}\ \phi_{js}\ [\bar{S}_r(\omega,\theta_r)\ \bar{S}_s(\omega,\theta_s)]^{1/2} \tag{57}$$

where θ is the phase angle of the excitation.

9. <u>From Modal Response to Physical Response</u>

In general $X = \Phi x$ or in component form

$$X_i = \sum_{j=1}^{n} \phi_{ij}\ x_j. \tag{58}$$

Consequently the mean physical response is

$$E[X_i] = \sum_{j=1}^{n} \phi_{ij}\ E[x_j] \tag{59}$$

and the mean square response is

$$E[X_i^{\,2}] = \sum_{j=1}^{n} \phi_{ij}^{\,2}\ E[x_j^{\,2}] + \sum_{\substack{j=1 \\ j\neq k}}^{n} \sum_{k=1}^{n} \phi_{ij}\ \phi_{ik}\ E[x_j x_k]. \tag{60}$$

if x_j and x_k are <u>independent</u> the quantity $E[x_j x_k] = \mathrm{cov}\ (x_j, x_k) = 0$, in which case

$$E[X_i^{\,2}] = \sum_{j=1}^{n} \phi_{ij}^{\,2}\ E[x_j^{\,2}]. \tag{61}$$

This situation occurs sufficiently often to be of interest in real applications. Finally,

$$\sigma_i^2 (X_i) = E[X_i^2] - \{E[X_i]\}^2. \tag{62}$$

Since the relationship between physical and modal acceleration is

$$\ddot{X}_i = \sum_{j=1}^{n} \phi_{ij} \ddot{x}_j$$

the steps outlined above are repeatable for the acceleration response of the linear system.

Other statistical considerations also play an important role in this Themis Project and therefore will be the subject of additional summary reports. These include fatigue damage criteria for structures under random excitation, the response of continuous (rather than lumped) structures to random excitation and the optimization of lumped parameter and continuous dynamic systems under random excitation.

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William F. Ames

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13. ABSTRACT

The dynamic environments to which weapon-vehicle systems are subjected include steady state harmonic excitation, shock and random excitation. These occur as a result of external stimuli such as atmospherically borne disturbances (wind, wake acoustic noise, rotor tip vortex loading) and through various weapon-vehicle interactions. In the problems of this project random excitation is the rule rather than the exception. Thus this report summarizes the probability techniques necessary for and their application to the development of analytic methods for obtaining the response of linear elastic structures to certain classes of random excitation.

This random vibration response analysis employs the normal modes of a lumped parameter representation of a complex system. The random forcing functions at each node (in terms of expected value and power spectral density) are transformed to a set of modal forcing functions. Then the response of each mode to a random forcing function can be obtained using the modal transfer functions. Finally the modal responses are transformed back to the physical plane. The results are the statistical expected values (mean, root mean square, power spectral density) of the displacements, velocities and accelerations of the physical system.

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